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SOME RESEARCH DIRECTIONS IN FINITE ELASTICITY THEORY.(U)

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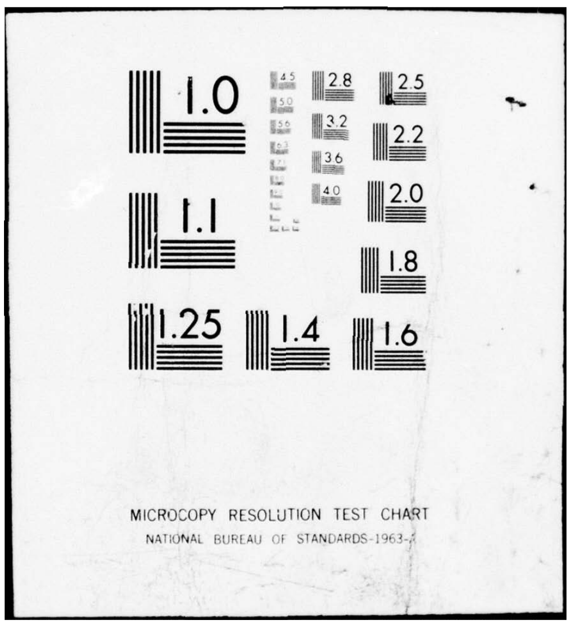
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SOME RESEARCH DIRECTIONS IN FINITE ELASTICITY THEORY

by

R.S. RIVLIN

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13. ABSTRACT

A number of topics in finite elasticity theory which appear to lend themselves to further development are briefly discussed. These include (i) the effect of kinematic constraints which are exactly, or approximately, satisfied; (ii) the mechanics of elastic membranes; (iii) the applicability of results in finite elasticity theory to problems involving stress relaxing materials; (iv) the development of necessary and sufficient conditions for material stability of isotropic elastic materials; (v) the conditions for bifurcation solutions to exist in deformed elastic bodies.

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Some Research Directions in Finite

Elasticity Theory*

by

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Abstract

A number of topics in finite elasticity theory which appear to lend themselves to further development are briefly discussed. These include (i) the effect of kinematic constraints which are exactly, or approximately, satisfied; (ii) the mechanics of elastic membranes; (iii) the applicability of results in finite elasticity theory to problems involving stress relaxing materials; (iv) the development of necessary and sufficient conditions for material stability of isotropic elastic materials; (v) the conditions for bifurcation solutions to exist in deformed elastic bodies.

*Keynote lecture delivered at Euromech Colloquium 79 on "Solutions to basic problems in non-linear continua", held in Darmstadt, German Federal Republic, September 7-10, 1976.

1. Introduction

The object of this paper is to present some research directions in finite elasticity theory, which have not, in the opinion of the author been fully exploited. The choice of topics has been governed by discipline oriented rather than problem oriented considerations. That is to say, the choice has been determined, not by the importance of the engineering applications which may stem from the researches discussed, but rather by a desire to indicate directions in which the theory of finite elasticity can, in its present state of development, be extended in a logical fashion.

With these objectives in mind, the basic equations of finite elasticity are succinctly presented in §2. Then, in §§3 and 5, the modifications to these equations which result from the existence of kinematic constraints on the possible deformations of the material are discussed. These constraints may be local in character (§3), or global (§5). In §6, we discuss the manner in which problems could be solved in situations in which the kinematic constraint conditions are only approximately satisfied.

In §4, we draw attention to the applicability, with minor modifications, of certain results of finite elasticity theory to problems involving viscoelastic solids with fading memory. In §7 we mention briefly some recent ideas on the mechanics of elastic membranes, due mainly to Skalak and his collaborators, which would appear to merit further exploitation.

Finally in §§8 and 9, we describe some recent results of Sawyers and Rivlin on material stability of an incompressible isotropic elastic material and on the conditions for a bifurcation from the pure homogeneous deformation of a thick plate of such a material to exist when it is subjected to a thrust.

2. The basic equations of finite elasticity theory

We consider the isothermal deformation of a body of elastic material. Some identifiable configuration of the body, generally, but not necessarily, that in which the material is unstressed, is taken as the reference configuration. We identify the particles of the body by their vector positions $\underset{\sim}{X}$, with respect to a fixed origin, in the reference configuration. As the body is deformed, the vector positions of its particles change to $\underset{\sim}{x}$, where $\underset{\sim}{x}$ depends on $\underset{\sim}{X}$ and the time t , thus:

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t) . \qquad [2.1]$$

We denote the components of $\underset{\sim}{X}$ and $\underset{\sim}{x}$ in a fixed rectangular cartesian coordinate system x by X_A $(A=1,2,3)$ and x_i $(i=1,2,3)$ respectively.

The strain-energy per unit mass, denoted W , is assumed to depend on the local deformation gradients $x_{i,A}$. For brevity, we will use the notation

$$\underset{\sim}{g} = ||g_{iA}|| , \quad g_{iA} = x_{i,A} . \qquad [2.2]$$

$\underset{\sim}{g}$ is called the <u>deformation</u> <u>gradient</u> <u>tensor</u>.

Since the superposition on the assumed deformation of a rigid rotation leaves W unchanged, it follows from rather simple mathematical considerations that W must be expressible in terms of the deformation gradients through the components in the system x of the symmetric tensor $\underset{\sim}{C}$, the Cauchy strain tensor, defined by

$$\underset{\sim}{C} = \underset{\sim}{g}^{\dagger}\underset{\sim}{g} , \qquad [2.3]$$

where the dagger denotes the transpose. Thus,

$$W = W(\underset{\sim}{C}) . \qquad [2.4]$$

Symmetry of the material in its reference state implies restrictions on the form of W . Thus, if $\underset{\sim}{S}$ denotes a generic transformation of the 3×3 matrix representation of the *symmetry group* $\underset{\sim}{S}$ *of the material*, W *must satisfy the* relation

$$W(\underset{\sim}{S}\underset{\sim}{C}\underset{\sim}{S}^{\dagger}) = W(\underset{\sim}{C}) . \qquad [2.5]$$

The group $\underset{\sim}{S}$ and correspondingly its 3×3 matrix representation, is either the full orthogonal group or a sub-group of it. *Equation [2.5] must be satisfied for all elements of the group* $\underset{\sim}{S}$.

In the case when $\underset{\sim}{S}$ is either the full or proper orthogonal group, corresponding respectively *to an isotropic material which is or is not centrosymmetric in the reference configuration, it follows that* W *must be expressible as a function of* I_1, I_2 and I_3 , where

$$I_1 = \mathrm{tr}\,\underset{\sim}{C} , \quad I_2 = \tfrac{1}{2}[(\mathrm{tr}\,\underset{\sim}{C})^2 - \mathrm{tr}\,\underset{\sim}{C}^2] , \qquad [2.6]$$
$$I_3 = \det \underset{\sim}{C} ,$$

thus:

$$W = W(I_1, I_2, I_3) . \qquad [2.7]$$

For deformations which are possible in a real material, $\det \underset{\sim}{g}$ (which physically is the ratio between the volumes of

a material element at time t and in the reference configuration) must be positive and it follows that

$$\det \underset{\sim}{g} = I_3^{\frac{1}{2}} . \qquad [2.8]$$

Accordingly, W is expressible as a function of I_1, I_2, $\det \underset{\sim}{g}$, thus:

$$W = W(I_1, I_2, \det \underset{\sim}{g}) . \qquad [2.9]$$

For materials which are transversely isotropic, the restrictions implied by eq. [2.5] were obtained by Ericksen and Rivlin (1). Those corresponding to the cases when $\underset{\sim}{S}$ is one or other of the crystallographic point groups were determined by Smith and Rivlin (2).

The Cauchy stress tensor $\underset{\sim}{\sigma}$ and Piola-Kirchhoff stress tensor $\underset{\sim}{\Pi}$ are given in terms of W by*

$$\underset{\sim}{\sigma} = \rho I_3^{-\frac{1}{2}} \underset{\sim}{g} \left(\frac{\partial W}{\partial \underset{\sim}{g}}\right)^{\dagger} , \quad \underset{\sim}{\Pi} = \rho \left(\frac{\partial W}{\partial \underset{\sim}{g}}\right)^{\dagger} , \qquad [2.10]$$

where ρ is the material density in the reference configuration. $\underset{\sim}{\sigma}$ and $\underset{\sim}{\Pi}$ must, of course, satisfy appropriate equations of motion which are

$$I_3^{\frac{1}{2}} \underset{\sim}{\nabla}_x \cdot \underset{\sim}{\sigma} + \rho \underset{\sim}{\phi} = \rho \ddot{\underset{\sim}{x}} , \quad \underset{\sim}{\nabla}_X \cdot \underset{\sim}{\Pi} + \rho \underset{\sim}{\phi} = \rho \ddot{\underset{\sim}{x}} , \qquad [2.11]$$

where $\underset{\sim}{\nabla}_x$ denotes the spatial gradient taken with respect to the configuration at time t and $\underset{\sim}{\nabla}_X$ denotes that taken with respect to the reference configuration.

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* By $\partial W/\partial \underset{\sim}{g}$, we mean the matrix with element $\partial W/\partial x_{i,A}$ in the ith row and Ath column.

The stress boundary conditions on $\underset{\sim}{\sigma}$ and $\underset{\sim}{\Pi}$ are

$$\underset{\sim}{f} = \underset{\sim}{\sigma}\underset{\sim}{n} \; , \quad \underset{\sim}{F} = \underset{\sim}{\Pi}^{\dagger}\underset{\sim}{N} \; , \tag{2.12}$$

where $\underset{\sim}{n}$ and $\underset{\sim}{N}$ are the unit normals to the boundary at time t and in the reference configuration respectively and $\underset{\sim}{f}$ and $\underset{\sim}{F}$ are the applied forces, per unit surface area measured at time t and in the reference configuration respectively.

If the material considered is isotropic, so that W may be written in the form given in eq. [2.7], then from eqs. [2.3], [2.6], [2.7], and [2.10] it follows that $\underset{\sim}{\sigma}$ is given by

$$\underset{\sim}{\sigma} = 2\rho I_3^{-\frac{1}{2}}[(W_1+I_1W_2)\underset{\sim}{c} - W_2\underset{\sim}{c}^2 + I_3W_3\underset{\sim}{\delta}] \; , \tag{2.13}$$

where W_α denotes $\partial W/\partial I_\alpha$ $(\alpha=1,2,3)$, $\underset{\sim}{\delta}$ denotes the unit tensor, and $\underset{\sim}{c}$ is the Finger strain tensor defined by

$$\underset{\sim}{c} = \underset{\sim}{g}\underset{\sim}{g}^{\dagger} \; . \tag{2.14}$$

Apart from the explicit forms for W for transversely isotropic materials and for materials with symmetries corresponding to one or other of the crystallographic groups, the results discussed so far were known very early in the present century.

If σ_i $(i=1,2,3)$ are the principal stresses and λ_i $(i=1,2,3)$ are the principal extension ratios, then I_1, I_2, I_3 may be expressed by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \; , \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \; , \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{2.15}$$

and

$$\sigma_i = 2\rho I_3^{-\frac{1}{2}}\{W_1\lambda_i^2 + (I_1-\lambda_i^2)W_2\lambda_i^2 + I_3W_3\} . \qquad [2.16]$$

3. Kinematic constraints and controllable deformations

If the material considered is incompressible, then the deformations which are possible in it are subject to the constraint

$$I_3 = 1 \qquad [3.1]$$

and correspondingly the expression for $\underset{\sim}{\sigma}$ in eq. [2.10] is replaced by

$$\underset{\sim}{\sigma} = \rho \underset{\sim}{g} \left(\frac{\partial W}{\partial \underset{\sim}{g}}\right)^{\dagger} - p\underset{\sim}{\delta} \,, \qquad [3.2]$$

where p is arbitrary if the deformation is specified, reflecting the fact that the superposition of an arbitrary hydrostatic pressure on the force system applied to the body leaves the deformation unaltered. In the case when the material is isotropic, eq. [2.8] becomes

$$W = W(I_1, I_2) \qquad [3.3]$$

and from eq. [3.2] we find that eq. [2.13] is replaced by (3)

$$\underset{\sim}{\sigma} = 2\rho[(W_1 + I_1 W_2)\underset{\sim}{c} - W_2 \underset{\sim}{c}^2] - p\underset{\sim}{\delta} \,. \qquad [3.4]$$

The effect of constraints on the expression for the Cauchy stress was investigated further by Ericksen and Rivlin (1). They considered the possibility that a number of constraints on the deformation of the form

$$f_\alpha(\underset{\sim}{g}) = 0 \quad (\alpha = 1, 2, \ldots) \qquad [3.5]$$

are simultaneously present in the material and showed that eq. [2.10] for the Cauchy stress must then be replaced by

$$\underset{\sim}{\sigma} = \rho I_3^{-\frac{1}{2}} \underset{\approx}{g} \left(\frac{\partial W}{\partial \underset{\approx}{g}} \right)^{\dagger} - \sum_{\alpha} p_\alpha I_3^{-\frac{1}{2}} \underset{\approx}{g} \left(\frac{\partial f_\alpha}{\partial \underset{\approx}{g}} \right)^{\dagger} , \qquad [3.6]$$

where the p's are arbitrary if the deformation is specified.

An example of such constraints is provided by the case when there are one or more directions of inextensibility in the undeformed material. If elements of length which are initially in the direction of the x_1 axis are inextensible, the corresponding constraint condition is

$$C_{11} = 1 . \qquad [3.7]$$

The case when there are two directions of inextensibility and in addition the material is incompressible was first studied by Adkins and Rivlin (4). This case and that in which the material is reinforced with only one set of cords was further studied by them and by a number of other workers, notably by Spencer, Pipkin and Rogers. Excellent accounts of much of this work is given in the monograph by Spencer (5) and the review article by Pipkin (6).

The constraint imposed by incompressibility is, as is now well-known, much more significant in its implications than might at first sight appear. For, it enables us to solve certain problems without making any specific assumptions regarding the strain-energy function than those implied by isotropy and incompressibility and expressed by eqs. [3.3] and [3.1]. Although these problems are admittedly simple ones they lend themselves to experimental realization and enable one to take the critical step of actually determining for a particular material - vulcanized rubber - the manner in which the strain-energy function depends on I_1 and I_2 and to verify

the predictive value of the strain-energy function so determined.

For example, if we consider a thin square sheet with thickness h and unit edge, it is fairly easy to show that the forces T_1 and T_2, which must be applied in order to deform it into a rectangle of dimensions λ_1 and λ_2, are given by (7)

$$\begin{aligned} T_1 &= 2\rho h\left(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2}\right)(W_1+\lambda_2^2W_2) , \\ T_2 &= 2\rho h\left(\lambda_2 - \frac{1}{\lambda_2^3\lambda_1^2}\right)(W_1+\lambda_1^2W_2) , \end{aligned} \qquad [3.8]$$

where

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} , \quad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2\lambda_2^2 . \qquad [3.9]$$

Experimentally, we can measure T_1 and T_2 for specified values of λ_1 and λ_2 and then use eqs. [3.8] to determine W_1 and W_2 for the corresponding values of I_1 and I_2 obtained from eqs. [3.9].

Again, we may consider the torsion of a circular cylinder, of radius a, by forces applied to its plane ends. If the twist per unit length is ψ, we find that in order to maintain the torsion we have to apply to the plane ends circumferential forces Θ per unit area and normal forces Z per unit area, given by

$$\begin{aligned} \Theta &= 2\psi r\rho(W_1+W_2) , \\ Z &= -2\psi^2\rho\left(r^2W_2 + \int_a^r rW_1 dr\right) , \end{aligned} \qquad [3.10]$$

where, in this case,

$$I_1 = I_2 = 3 + \psi^2 r^2 \ . \qquad [3.11]$$

Deformations which can be supported without application of body forces and for which the surface forces can be calculated without making specific assumptions regarding the form of the strain-energy function, beyond those implied by such material symmetry and kinematic constraints as may be relevant to the class of materials under consideration, are called <u>controllable</u> <u>deformations</u>. The attempt to find <u>all</u> possible controllable deformations in an isotropic material was initiated by Ericksen (8) and considerably extended by Singh and Pipkin (9) and by Klingbeil and Shield (10).

In the case of compressible isotropic elastic materials, it was shown by Ericksen (11) that the class of controllable deformations consists only of homogeneous deformations. In the case of incompressible materials it is substantially wider.

We may also identify a class of deformations which may be called <u>quasi</u>-<u>controllable</u> <u>deformations</u>. As an example of these, we may consider (12) an incompressible isotropic elastic material to be contained in the annular region between two rigid coaxial circular cylinders, one of which is moved parallel to the other by a fixed amount, or rotated relative to the other, about the common axis, through a specified angle. In either case the displacements of the particles of the material are determined as a function of radial position r by a single ordinary differential equation which depends on the form of W. The surface forces which must be applied in order to maintain the

deformation can then be calculated leaving open the precise form of the strain-energy function and the consequent radial distribution of the displacement.

Another type of system which leads to tractable problems is the elastic membrane - typically a vulcanized rubber membrane - which can be subjected to large deformations. Work on such systems was first carried out by Adkins and Rivlin (13). If the membrane and the deforming forces have rotational symmetry about an axis, the local deformation is a pure homogeneous deformation with principal directions along the lines of longitude and latitude in the deformed membrane and principal extension ratios in the surface of the membrane, λ_1, λ_2 say. The corresponding two-dimensional Piola-Kirchhoff stress is then given by eqs. [3.8]. If W is any known function of I_1 and I_2, the point equations of equilibrium can be integrated numerically.

4. An application of finite elasticity theory to inelastic materials

It was noted by Rivlin (14) that the expressions obtained for the forces which must be applied to maintain controllable deformations in isotropic* elastic materials may be applied, with changes which are, from a mathematical standpoint, trivial in nature, to obtain expressions for the forces which must be applied in order to maintain identical deformations in stress-relaxing materials.

We consider that a body of stress-relaxing material is deformed during a small time interval Δt_o, say, at the instant of time t_o, say, and that the deformation is then held constant. We assume that after the lapse of a sufficient time, the Cauchy stress $\underset{\sim}{\sigma}$ depends on the deformation gradient $\underset{\sim}{g}$ and the time that has elapsed**, t say, after the deformation is produced. Thus,

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{g}, t) . \qquad [4.1]$$

It can easily be shown, from the consideration that if an arbitrary rigid rotation is superposed on the assumed deformation, the Cauchy stress tensor is rotated by an equal amount, that $\underset{\sim}{\sigma}$ must be expressible in the form

$$\underset{\sim}{\sigma} = \underset{\sim}{g}\underset{\sim}{f}(\underset{\sim}{C}, t)\underset{\sim}{g}^{\dagger} , \qquad [4.2]$$

*Similar considerations apply, in principle, to stress-relaxing anisotropic materials.

**The extent to which this assumption is valid depends, of course, on the nature of the stress-relaxing material and on the class of allowable deformation histories by which the final state of static deformation is attained.

where $\underset{\sim}{C}$ is the Cauchy strain defined by eq. [2.3]. This step in our argument parallels that by which the passage was achieved from the assumption for an elastic material that W is a function of $\underset{\sim}{g}$ to the result expressed by eq. [2.4]. Again, paralleling the passage from eq. [2.4] to eq. [2.5], we find that if the material has symmetry in its reference state, then with the notation of §2

$$f(\underset{\sim}{S}\underset{\sim}{C}\underset{\sim}{S}^{\dagger}) = \underset{\sim}{S}\underset{\sim}{f}(\underset{\sim}{C})\underset{\sim}{S}^{\dagger} \ . \qquad [4.3]$$

For an isotropic material this leads to the conclusion that $\underset{\sim}{\sigma}$ must be expressible in the form

$$\underset{\sim}{\sigma} = \alpha_1\underset{\sim}{c} + \alpha_2\underset{\sim}{c}^2 + \alpha_0\underset{\sim}{\delta} \ , \qquad [4.4]$$

where the α's are functions of I_1, I_2 and I_3 and of t and $\underset{\sim}{c}$ is the Finger strain defined by eq. [2.14]. For an incompressible material, this relation becomes

$$\underset{\sim}{\sigma} = \alpha_1\underset{\sim}{c} + \alpha_2\underset{\sim}{c}^2 - p\underset{\sim}{\delta} \ , \qquad [4.5]$$

where the α's are now functions of I_1, I_2 and t and p is an arbitrary hydrostatic pressure. Comparing this equation with eq. [3.4], we see that the latter can be obtained from eq. [4.5] by replacing α_1 and α_2 by $2\rho(W_1+I_1W_2)$ and $-2\rho W_2$ respectively.

The forces necessary to maintain a controllable deformation is a stress-relaxing material can be obtained from those for an elastic material, by replacing W_1 and W_2 by $(\alpha_1+I_1\alpha_2)/2\rho$ and $-\alpha_2/2\rho$ respectively. For example, the formulae [3.10] for the azimuthal and normal forces, Θ and Z per unit area, which

must be applied to the plane ends of a right-circular cylinder in order to maintain in it a simple torsion of amount ψ become

$$\Theta = \psi r[\alpha_1 + (2+\psi^2 r^2)\alpha_2] \; ,$$

$$Z = -\psi^2 \left[-\alpha_2 r^2 + \int_a^r r\{\alpha_1 + (3+\psi^2 r^2)\alpha_2\}dr\right] \; , \qquad [4.6]$$

where the α's are functions of $\psi^2 r^2$ and of elapsed time after the deformation is produced.

5. Global constraints

A somewhat different type of constraint from those discussed in §3 may be appropriate to certain systems. For example, if a body of vulcanized rubber is swollen homogeneously in some solvent and is then inhomogeneously deformed, the solvent may migrate from one part of the body to another. Since both the vulcanized rubber and the solvent separately may, with good approximation, be treated as incompressible, the total volume of the body will be substantially unaltered by the deformation, but each element of the vulcanized rubber may change the volume it occupies due to the absorption or loss of solvent. Accordingly, I_3, defined in eqs. [2.6], may be regarded as a measure of the volume fraction of solvent at any point and the strain-energy, at any rate for quasi-static deformations of the swollen rubber vulcanizate must be expressible in the form [2.7]. The constraint implied by the constancy of volume of the body may, of course, be written as

$$\int_V \det \underset{\sim}{g} \cdot dV = V \,, \qquad [5.1]$$

where V is the domain occupied by the body in the homogeneous reference configuration. The point equations which can be derived for the type of system considered are similar to those for a compressible material, except that the force boundary condition is now given by

$$t_i = (\sigma_{ij} + p\delta_{ij})n_j \qquad [5.2]$$

and the constraint condition by eq. [5.1].

6. Perturbations of the constraint condition

We note that if we superpose on the deformation of a body described by eq. [2.1], a uniform dilatation, so that $\underset{\sim}{x}$ is replaced by $\lambda\underset{\sim}{x}$, where λ is independent of $\underset{\sim}{X}$, then the Finger strain $\underset{\sim}{c}$ defined by eq. [2.14] is replaced by $\lambda^2\underset{\sim}{c}$. Similarly, the strain invariants I_1, I_2 and I_3 , defined by eqs. [2.6], are replaced by $\lambda^2 I_1$, $\lambda^4 I_2$ and $\lambda^6 I_3$ respectively. Thus, $\bar{\underset{\sim}{c}}$, J_1 and J_2 defined by

$$\bar{\underset{\sim}{c}} = I_3^{-1/3}\underset{\sim}{c} , \quad J_1 = I_1 I_3^{-1/3} , \quad J_2 = I_2 I_3^{-2/3} \qquad [6.1]$$

are unaltered by the superposed dilatation. This suggests (Flory (15), Penn (16)) that, for an isotropic material, we express W as a function of J_1, J_2 and I_3 , thus

$$W = W(J_1, J_2, I_3) , \qquad [6.2]$$

rather than as a function of I_1, I_2, I_3 as indicated in eq. [2.7]. More conveniently for our purposes, we can express W as a function of J_1, J_2 and τ $(=I_3^{1/2})$, thus:

$$W = W(J_1, J_2, \tau) . \qquad [6.3]$$

Then, the expression [2.13] for $\underset{\sim}{\sigma}$ can be rewritten as

$$\underset{\sim}{\sigma} = \frac{2\rho}{\tau}\left\{\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2}\right)\bar{\underset{\sim}{c}} - \frac{\partial W}{\partial J_2}\bar{\underset{\sim}{c}}^2 + \left[-\frac{1}{3}J_1\frac{\partial W}{\partial J_1} - \frac{2}{3}J_2\frac{\partial W}{\partial J_2} + \frac{1}{2}\tau\frac{\partial W}{\partial \tau}\right]\underset{\sim}{\delta}\right\}. \qquad [6.4]$$

If W is an analytic function of τ in the neighborhood of any isochoric deformation, it can be approximated for deformations for which the dilatation $\tau-1$ is sufficiently small by

$$W = w_0(J_1,J_2) + (\tau-1)w_1(J_1,J_2) + \frac{1}{2}(\tau-1)^2 w_2(J_1,J_2) , \qquad [6.5]$$

where

$$w_0(J_1,J_2) = W\Big|_{\tau=1} , \quad w_1(J_1,J_2) = \frac{\partial W}{\partial \tau}\Big|_{\tau=1} ,$$
$$w_2(J_1,J_2) = \frac{\partial^2 W}{\partial \tau^2}\Big|_{\tau=1} . \qquad [6.6]$$

Then, if we introduce eq. [6.5] into eq. [6.4] and neglect terms of higher degree than the first in $\tau-1$, we obtain

$$\begin{aligned}\underset{\sim}{\sigma} = 2\rho&\left\{\left(\frac{\partial w_0}{\partial J_1} + J_1\frac{\partial w_0}{\partial J_2}\right)\underset{\sim}{\bar{c}} - \frac{\partial w_0}{\partial J_2}\underset{\sim}{\bar{c}}^2\right.\\ &\left.- \left(\frac{1}{3}J_1\frac{\partial w_0}{\partial J_1} + \frac{2}{3}J_2\frac{\partial w_0}{\partial J_2} - \frac{1}{2}w_1\right)\underset{\sim}{\delta}\right\}\\ &+ 2\rho(\tau-1)\left\{\left(\frac{\partial w_1}{\partial J_1} + J_1\frac{\partial w_1}{\partial J_2} - \frac{\partial w_0}{\partial J_1} - J_1\frac{\partial w_0}{\partial J_2}\right)\underset{\sim}{\bar{c}}\right.\\ &- \left(\frac{\partial w_1}{\partial J_2} - \frac{\partial w_0}{\partial J_2}\right)\underset{\sim}{\bar{c}}^2 + \left[\frac{1}{3}J_1\left(\frac{\partial w_0}{\partial J_1} - \frac{\partial w_1}{\partial J_1}\right)\right.\\ &\left.\left.+ \frac{2}{3}J_2\left(\frac{\partial w_0}{\partial J_2} - \frac{\partial w_1}{\partial J_2}\right) + \frac{1}{2}w_2\right]\underset{\sim}{\delta}\right\} . \qquad [6.7]\end{aligned}$$

We consider a sequence of constitutive equations for nearly compressible materials, with specified density ρ in the undeformed state, which has as its limit the constitutive equation for an incompressible material with strain-energy function given by

$$W = w_0(J_1,J_2) . \qquad [6.8]$$

The corresponding expression for the stress is

$$\underset{\sim}{\sigma} = 2\rho\left\{\left(\frac{\partial w_0}{\partial J_1} + J_1\frac{\partial w_0}{\partial J_2}\right)\underset{\sim}{\bar{c}} - \frac{\partial w_0}{\partial J_2}\underset{\sim}{\bar{c}}^2\right\} - p\underset{\sim}{\delta} , \qquad [6.9]$$

where p is an arbitrary hydrostatic pressure. We note that since all deformations in an incompressible material are necessarily isochoric, we have $\bar{\underset{\sim}{c}}=\underset{\sim}{c}$ in this case.

The fact that for the incompressible material the constitutive equation for $\underset{\sim}{\sigma}$ must take the form [6.9] leads to the restriction on the coefficient of $2\rho(\tau-1)$ in eq. [6.7] that it tends to infinity, in such a way that its product with $(\tau-1)$ remains finite, as the material tends to incompressibility.

We can solve problems for slightly compressible materials, for which eqs. [6.5] and [6.7] are valid, by first solving the problem for an incompressible material for which the eqs. [6.8] and [6.9] are valid. Then, using a perturbation procedure, we can find the correction to the displacement field appropriate to the slightly compressible material.

Analogous procedures should be possible in situations where other constraint conditions than incompressibility are approximately valid. However, it must be borne in mind that the perturbations involved need not necessarily be regular (Rogers and Pipkin (17), Everstine and Pipkin (18)).

7. Two-dimensional materials

One can construct an interesting two-dimensional analog to finite elasticity theory, which is apparently applicable to various living tissues, e.g. the membrane of the human red blood cell (Hochmuth and Mohandas (19), Skalak, Tozeren, Zarda and Chien (20)).

We consider a membrane of transversely isotropic elastic material, the axis of rotational symmetry being normal to the membrane. With the usual assumptions of membrane theory, we can write the strain-energy per unit area (measured in the reference state) as a function of I_1, I_2 defined in terms of the principal extension ratios λ_1, λ_2 by

$$I_1 = \lambda_1^2 + \lambda_2^2 , \quad I_2 = \lambda_1^2\lambda_2^2 , \qquad [7.1]$$

thus:

$$W = W(I_1, I_2) . \qquad [7.2]$$

The principal line stresses σ_α $(\alpha=1,2)$ are then given by

$$\sigma_\alpha = 2I_2^{-\frac{1}{2}}\left(\lambda_\alpha^2 \frac{\partial W}{\partial I_1} + I_2 \frac{\partial W}{\partial I_2}\right) . \qquad [7.3]$$

It appears that for certain tissues, the area of an element of the membrane reamins approximately constant as the membrane is deformed. We are therefore led to construct a constitutive equation which idealizes this observation by imposing the constraint condition*

$$I_2 = 1 \qquad [7.4]$$

---

*If the material of the membrane is also incompressible, this implies, of course, that it is inextensible in a direction normal to it.

on the deformation. Then, W is a function of I_1 only, thus:

$$W = W(I_1) \qquad [7.5]$$

and the constitutive equation [7.3] is replaced by

$$\sigma_\alpha = 2\lambda_\alpha^2 \frac{\partial W}{\partial I_1} - p \ , \qquad [7.6]$$

where p is arbitrary if the deformation is specified.

If the constraint [7.4] is only approximate, the resulting change in the deformation could be calculated in a manner analogous to that introduced in the case of slightly compressible materials.

8. Initial stress problems

We have seen that for incompressible isotropic elastic materials, certain simple problems can be solved without particularizing the form of the strain-energy function beyond that given in eq. [3.3]. The recognition of this fact led to the formulation of a rational theory for the solution of initial stress problems in which the strain-energy function is left in the form [3.3] and the initial stress is that corresponding to one or other of the so-called controllable solutions. Insofar as controllable solutions exist for compressible isotropic materials, analogous considerations apply.

The Piola-Kirchhoff stress $\underset{\sim}{\Pi}$ for a compressible material corresponding to a deformation $\underset{\sim}{X} \to \underset{\sim}{x}$ is given by eq. [2.10], viz:

$$\Pi_{Ai} = \rho \frac{\partial W}{\partial x_{i,A}} . \qquad [8.1]$$

We now consider an infinitesimal superposed deformation $\underset{\sim}{x} \to \underset{\sim}{x} + \varepsilon \underset{\sim}{u}$, where ε is sufficiently small so that we can linearize in it. This superposed deformation may be time-independent or time-dependent. The incremental stress $\varepsilon \pi_{Ai}$ is then given by

$$\pi_{Ai} = \rho \frac{\partial^2 W}{\partial x_{i,A} \partial x_{j,B}} u_{j,B} . \qquad [8.2]$$

If the material is isotropic, we can introduce the expression [2.7] for W and carry out the indicated differentiations to obtain the appropriate expression for π_{Ai} . In this expression we substitute for the coefficients of $u_{j,B}$, the expressions

appropriate to the known underlying static deformation. The appropriate equation of motion for the superposed deformation is

$$\pi_{Ai,A} = \rho \ddot{u}_i \qquad [8.3]$$

and the force boundary condition is

$$\pi_{Ai} N_A = \bar{f}_i \ , \qquad [8.4]$$

where $\epsilon \bar{f}_i$ is the additional force associated with the superposed deformation.

In the case when the material is incompressible, the expression [8.2] for the incremental stress must be modified to reflect the fact that the superposition of a hydrostatic pressure does not change the deformation. In addition, we have to add an equation which expresses the fact that no volume changes can take place.

On the basis of the considerations outlined above, a number of initial stress problems have been solved. The earliest of these was the calculation by Green and Shield (21) of the effect of an initial finite extension on the torsional modulus, for infinitesimal torsion, of a rod of arbitrary but uniform cross-section. This generalized an earlier result of Rivlin (12) for a rod of circular cross-section. The result of Green and Shield (21) was corrected and verified experimentally by Gent and Rivlin (22). Shortly afterwards, Green, Rivlin and Shield (23) discussed the effect of initial stress on the indentation by a punch of an elastic half-space subjected to an initial biaxial pure homogeneous deformation with two

equal principal extension ratios which define a principal plane parallel to the bounding plane of the half-space. These results and many others of a similar character have been discussed extensively in the subsequent literature. Here we shall consider only the initial stress problem for plane waves of infinitesimal amplitude propagating in a deformed elastic material.

First, let us suppose that the medium in which the plane wave propagates is a space of isotropic elastic material subjected to a pure homogeneous deformation with principal extension ratios $\lambda_1, \lambda_2, \lambda_3$ and principal directions parallel to the axes of a rectangular cartesian coordinate system x . Let us suppose also that the wave is sinusoidal, has angular frequency ω , wave number k , slowness $S = k/\omega$, and propagates in the direction of the unit vector $\underset{\sim}{n}$. Then, with the usual complex notation, we can express the displacement vector $\varepsilon \underset{\sim}{u}$ associated with this wave in the form

$$\underset{\sim}{u} = \underset{\sim}{A} \exp \iota(k\underset{\sim}{n}\cdot\underset{\sim}{x} - \omega t) . \qquad [8.5]$$

The secular equation determining the complex slowness S was obtained by Hayes and Rivlin (24) in the case when the material is compressible. Here, however, we shall give the somewhat simpler results, obtained by Sawyers and Rivlin (25), for the case when the material is incompressible. In this case the secular equation is

$$\alpha S^4 - \beta\rho S^2 + \rho^2 = 0 , \qquad [8.6]$$

where

$$\alpha = 4(\lambda_1^2 n_1^2 + \ldots)\{[\lambda_1^2 n_1^2 (W_1 + \lambda_2^2 W_2)(W_1 + \lambda_3^2 W_2) + \ldots]$$
$$+ W_1 [n_2^2 n_3^2 (\lambda_2 - \lambda_3)^2 (W_1 + \lambda_1^2 W_2) A_1 + \ldots]\}$$
$$+ 4W_2 [\lambda_2^2 \lambda_3^2 n_2^2 n_3^2 (\lambda_2 - \lambda_3)^2 (W_1 + \lambda_1^2 W_2) A_1 + \ldots]$$
$$- 16 n_1^2 n_2^2 n_3^2 (\lambda_2^2 - \lambda_3^2)^2 (\lambda_3^2 - \lambda_1^2)^2 (\lambda_1^2 - \lambda_2^2)^2 (W_{12}^2 - W_{11} W_{22}) ,$$

$$\beta = 2(W_1 + \lambda_1^2 W_2)[\lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 + (\lambda_2 - \lambda_3)^2 n_2^2 n_3^2 A_1] + \ldots , \qquad [8.7]$$

where the dots represent terms obtained from those given by cyclic permutation of the subscripts on the λ's, n's and A's and where the A's are defined by

$$A_1 = 2(\lambda_2 + \lambda_3)^2 \frac{W_{11} + 2\lambda_1^2 W_{12} + \lambda_1^4 W_{22}}{W_1 + \lambda_1^2 W_2} , \qquad [8.8]$$

with analogous expressions for A_2 and A_3. W_{11}, W_{12} and W_{22} denote $\partial^2 W/\partial I_1^2$, $\partial^2 W/\partial I_1 \partial I_2$ and $\partial^2 W/\partial I_2^2$ respectively.

We may regard eq. [8.6] as a quadratic equation in S^2. The necessary and sufficient conditions for it to yield two positive roots for S^2 are

$$\alpha > 0 , \quad \beta > 0 , \quad \beta^2 \geq 4\alpha . \qquad [8.9]$$

Following Hadamard, we adopt these conditions as the conditions for material stability.

It would be of interest to obtain conditions on W such that the relations [8.9] are satisfied for all $\tilde{n}$, i.e. for all directions of the normal to the wave-front. So far this has not been achieved. However, Sawyers and Rivlin (25) have obtained necessary and sufficient conditions on W, so that the relations [8.9] are valid for all waves for which $\tilde{n}$ lies

in a principal plane of the pure homogeneous deformation. In this case necessary and sufficient conditions that the relations [8.9] be satisfied are

$$W_1 + \lambda_i^2 W_2 > 0 \ ,$$
$$W_{11} + 2\lambda_i^2 W_{12} + \lambda_i^4 W_{22} > -\frac{1}{2}\left(I_1 - \lambda_i^2 - \frac{2}{\lambda_i}\right)^{-1}(W_1 + \lambda_i^2 W_2) \qquad [8.10]$$
$$(i=1,2,3) \ .$$

The conditions $[8.10]_1$ were previously obtained by Baker and Ericksen (26) and are usually referred to in the literature as the Baker-Ericksen conditions. It has also been shown by Sawyers and Rivlin, in unpublished work, that the condition $\beta > 0$ follows from the Baker-Ericksen conditions.

The conditions [8.10] can also be obtained (27) from the following considerations. We consider the material to be subjected to a pure homogeneous deformation as before. We now superpose an infinitesimal simple shear for which the plane of shear is a principal plane of the pure homogeneous deformation and the direction of shear is that of a unit vector, $\underset{\sim}{K}$ say, lying in this plane. We consider the change, as a result of this superposed simple shear, in the traction in the direction of $\underset{\sim}{K}$ acting on unit area of a plane parallel to $\underset{\sim}{K}$ and to the normal to the principal plane considered. The ratio between this traction and the amount of shear is the incremental shear modulus. The relations [8.10] are necessary and sufficient conditions for this to be positive for all choices of $\underset{\sim}{K}$ in a principal plane.

9. Bifurcation solutions for a thick plate under thrust

An example of another type of initial stress problem which can be solved, without imposing restrictions on the form of the strain-energy function beyond those implied by isotropy and incompressibility, is provided by the calculation of the bifurcation condition for a thick rectangular plate of isotropic incompressible elastic material, subjected to biaxial loading.

Suppose the edges of the plate are parallel to the axes of a rectangular cartesian coordinate system x and that the lengths of these edges are $2\ell_1$, $2\ell_2$, $2\ell_3$ $(2\ell_3 >> 2\ell_1, 2\ell_2)$. The plate is subjected to a pure homogeneous deformation, with extension ratios λ_1, λ_2, λ_3 and principal directions parallel to the axes of the system x , and is held so that λ_3 remains fixed, the faces perpendicular to the 2-direction being force-free. The values of λ_1, λ_2 are then varied (in accordance, of course, with the incompressibility condition $\lambda_1\lambda_2 = \lambda_3^{-1}$) and the critical value of λ $(= \lambda_2/\lambda_1)$ is determined for which a bifurcation of the flexural or barreling type in the 12-plane can exist. The faces initially perpendicular to the 1-direction are constrained, so that they remain perpendicular to the 1-direction and the tangential surface traction is zero over them. While these constraints are admittedly somewhat artificial, they have the merit that, with them, the problem can be solved exactly.

We denote the wave-length of the superposed flexural or barreling deformation by $2\pi/\Omega$, so that

$$\Omega = n\pi/2\ell_1 \quad (n=1,2,\ldots) . \qquad [9.1]$$

Then, the critical value of λ, at which a bifurcation from the state of pure homogeneous deformation becomes possible, is given, for specified values of n (i.e. of the mode of the superposed flexure or barreling) and of ℓ_2/ℓ_1, by

$$\frac{\tanh \Omega_2\ell_2}{\tanh \Omega_1\ell_2} = \left(\frac{\Omega_2^2+\lambda^2\Omega^2}{\Omega_1^2+\lambda^2\Omega^2}\right)^2 \frac{\Omega_1}{\Omega_2},$$

or [9.2]

$$\frac{\tanh \Omega_1\ell_2}{\tanh \Omega_2\ell_2} = \left(\frac{\Omega_2^2+\lambda^2\Omega^2}{\Omega_1^2+\lambda^2\Omega^2}\right)^2 \frac{\Omega_1}{\Omega_2},$$

accordingly as the superposed deformation is of the flexural or barreling type. Ω_1 and Ω_2 are defined by

$$\Omega_1^2,\Omega_2^2 = \frac{1}{2}\,\Omega^2\{[1+\lambda^2+A(1-\lambda)^2] \pm [\{1+\lambda^2+A(1-\lambda)^2\}^2-4\lambda^2]^{\frac{1}{2}}\}, \quad [9.3]$$

where A is defined by

$$A = \frac{2(\lambda_1+\lambda_2)^2}{W_1+\lambda_3^2W_2}\,(W_{11}+2\lambda_3^2W_{12}+\lambda_3^4W_{22}). \quad [9.4]$$

Equations [9.2] were apparently first derived by Wesolowski (28).

It follows from eq. [9.3] that formally, at any rate, we may discuss the implications of eqs. [9.2] in three distinct cases:

(i) Ω_1^2 and Ω_2^2 are both positive,

(ii) Ω_1^2 and Ω_2^2 are complex conjugates,

(iii) Ω_1^2 and Ω_2^2 are both negative.

It is apparent from eq. [9.3] that case (iii) arises if and only if

$$A < -\left(\frac{\lambda+1}{\lambda-1}\right)^2 , \qquad [9.5]$$

case (ii) arises if and only if

$$-\left(\frac{\lambda+1}{\lambda-1}\right)^2 \leq A < -1 , \qquad [9.6]$$

and case (i) arises if and only if

$$A \geq -1 . \qquad [9.7]$$

The necessary conditions [8.10] for material stability derived earlier imply that we need consider only cases (i) and (ii). We shall, in fact, consider only case (i).

In order to discuss the qualitative implications of eqs. [9.2], Sawyers and Rivlin (29) found it convenient to rewrite them in the form

$$\frac{\sinh[(2\lambda^{\frac{1}{2}}\cosh\,\delta)\eta]}{\sinh[(2\lambda^{\frac{1}{2}}\sinh\,\delta)\eta]} = \pm\,\frac{[(1+\lambda^2)^2+4\,\sinh^2\delta]\cosh\,\delta}{[-(1-\lambda)^2+4\,\cosh^2\delta]\sinh\,\delta} , \qquad [9.8]$$

where

$$\eta = \ell_2\Omega = \frac{n\pi\ell_2}{2\ell_1} \quad \text{and} \quad \sinh^2\delta = \frac{1}{4\lambda}\,(1+A)(\lambda-1)^2 \qquad [9.9]$$

and the plus and minus signs in eqs. [9.8], apply to superposed deformations of the flexural and barreling types respectively.

Equations [9.8] may be regarded as equations for the determination of $\ell_2\Omega$, and hence of $n\ell_2/\ell_1$, corresponding to a specified critical value of λ . Then, provided $A > -1$,

i.e., δ is real, it is fairly easy to prove the following theorems:

(a) If $\lambda \leq 1$, corresponding to the situation when the applied force in the 1-direction is tensile or zero, neither of the eqs. [9.8] has a real solution for $n\ell_2/\ell_1$. Accordingly in this case no bifurcation of either the flexural or barreling type is possible.

(b) If $\lambda > 1$, corresponding to the situation when the applied force in the 1-direction is a thrust, and the upper sign in eqs. [9.8] is taken, so that we are considering only superposed deformations of the flexural type, then one and only one solution for $n\ell_2/\ell_1$ exists provided that

$$4 \cosh^2\delta - (\lambda-1)^2 > 0 \qquad [9.10]$$

and no solution exists if

$$4 \cosh^2\delta - (\lambda-1)^2 \leq 0 \,. \qquad [9.11]$$

(c) If $\lambda > 1$ and the lower sign is taken in eqs. [9.8], so that we are considering superposed deformations of the barreling type, then one and only one solution for $n\ell_2/\ell_1$ exists provided that

$$4 \cosh^2\delta - (\lambda-1)^2 < 0 \qquad [9.12]$$

and no solution exists if

$$4 \cosh^2\delta - (\lambda-1)^2 \geq 0 \,. \qquad [9.13]$$

It follows that whatever may be the strain-energy function, subject to the restriction $A > -1$, the ranges of critical values of λ at which bifurcations of the flexural and barreling types

can occur are discrete. The value of λ separating these regions is, of course, given by

$$4 \cosh^2\delta - (\lambda-1)^2 = 0 \ , \qquad [9.14]$$

and, with eq. $[9.9]_2$, by

$$\lambda^3 - (3+A)\lambda^2 + (2A-1)\lambda - (1+A) = 0 \ . \qquad [9.15]$$

We bear in mind here that if the expression for W as a function of I_1 and I_2 is known then the dependence of A on λ is also known.

For any constant value of A we can plot η against the critical value of λ for a bifurcation to occur. We obtain curves of the form shown in Fig. 1. Generally, the situation A = constant is somewhat artifical*. However, curves such as those in Fig. 1 can be used to determine the critical values for *bifurcations of the flexural and barreling types*, corresponding to <u>any</u> specified strain-energy function and a specified value of η, and hence of $n\ell_2/\ell_1$. From Fig. 1 we read off, for the specified value of η, corresponding values of A and λ, for flexure, *say*. We plot these on a graph and on the same graph we plot corresponding values of A and λ obtained from the expression [9.4] for A with the specified strain-energy function. The intersection of these two curves gives us the critical value of λ for a bifurcation of the flexural type to exist. An analogous procedure using A vs. λ curves for barreling, obtained from Fig. 1, enables us to determine the

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*An exception arises in the case of the Mooney-Rivlin strain-energy function for which $A=0$.

critical value of λ for a bifurcation of the barreling type to occur.

A particularly interesting conclusion results in the case when η , and hence $n\ell_2/\ell_1$, is small (30). Using eq. [9.8] with the plus sign, we can obtain the following formula for the critical value of η , corresponding to bifurcation of the flexural type, as a function of η , in the form

$$\lambda - 1 = a_1\eta^2 + a_2\eta^4 + a_3\eta^6 + a_4\eta^8 + O(\eta^{10}) \ , \qquad [9.16]$$

where

$$a_1 = \frac{2}{3} \ , \quad a_2 = \frac{16}{45} \ , \quad a_3 = \frac{2}{27}\left(A_0 + \frac{18}{7}\right) \ ,$$
$$a_4 = \frac{2}{27}\left(A_0 + \frac{113}{175}\right) \ , \qquad [9.17]$$

and

$$A_0 = \mathop{\mathrm{Lt}}_{\lambda \to 1} A \ . \qquad [9.18]$$

It is easy to show that whatever may be the expression for W ,

$$A = A_0 + O(\lambda - 1)^2 = A_0 + O(\eta^4) \ . \qquad [9.19]$$

We see from this result that the critical value of $\lambda - 1$ is independent of A_0 up to terms of order η^4 . Also, it depends on W only through its second derivative, evaluated in the state for which the thrust is zero, up to terms of order η^8 , so that for all materials having a strain-energy of the Mooney-Rivlin form, the critical value of $\lambda - 1$ is the same up to terms of order η^8 .

To give some quantitative feel for the insentivity of the

critical value of λ to the value of η and hence of ℓ_2/ℓ_1, we mention that if $A_0=0$ and $\eta = .793$, which for $n=1$ corresponds to $\ell_2/\ell_1 = 0.505$, the value of λ obtained by taking the first three terms in eq. [9.16] is 1.608 and that obtained by using the exact formula [9.8] is 1.600, corresponding, if $\lambda_3=1$, to compressions of 21.14 and 20.94 per cent respectively.

A parallel insensitivity of the compressive force to the precise form of the strain-energy function is also found. The compressive force F, per unit cross-sectional area measured in the undeformed state, is given by

$$F = -4\lambda_3^{-\frac{1}{2}}T_0(a_1\eta^2+a_2\eta^4+b\eta^6+c\eta^8) + O(\eta^{10}) , \qquad [9.20]$$

where

$$b = \frac{1}{9}\left(A_0+\frac{43}{21}\right) , \quad c = \frac{34}{405}\left(A_0+\frac{584}{595}\right) , \qquad [9.21]$$

and

$$T_0 = (W_1+\lambda_3^2W_2)\big|_{\lambda=1} . \qquad [9.22]$$

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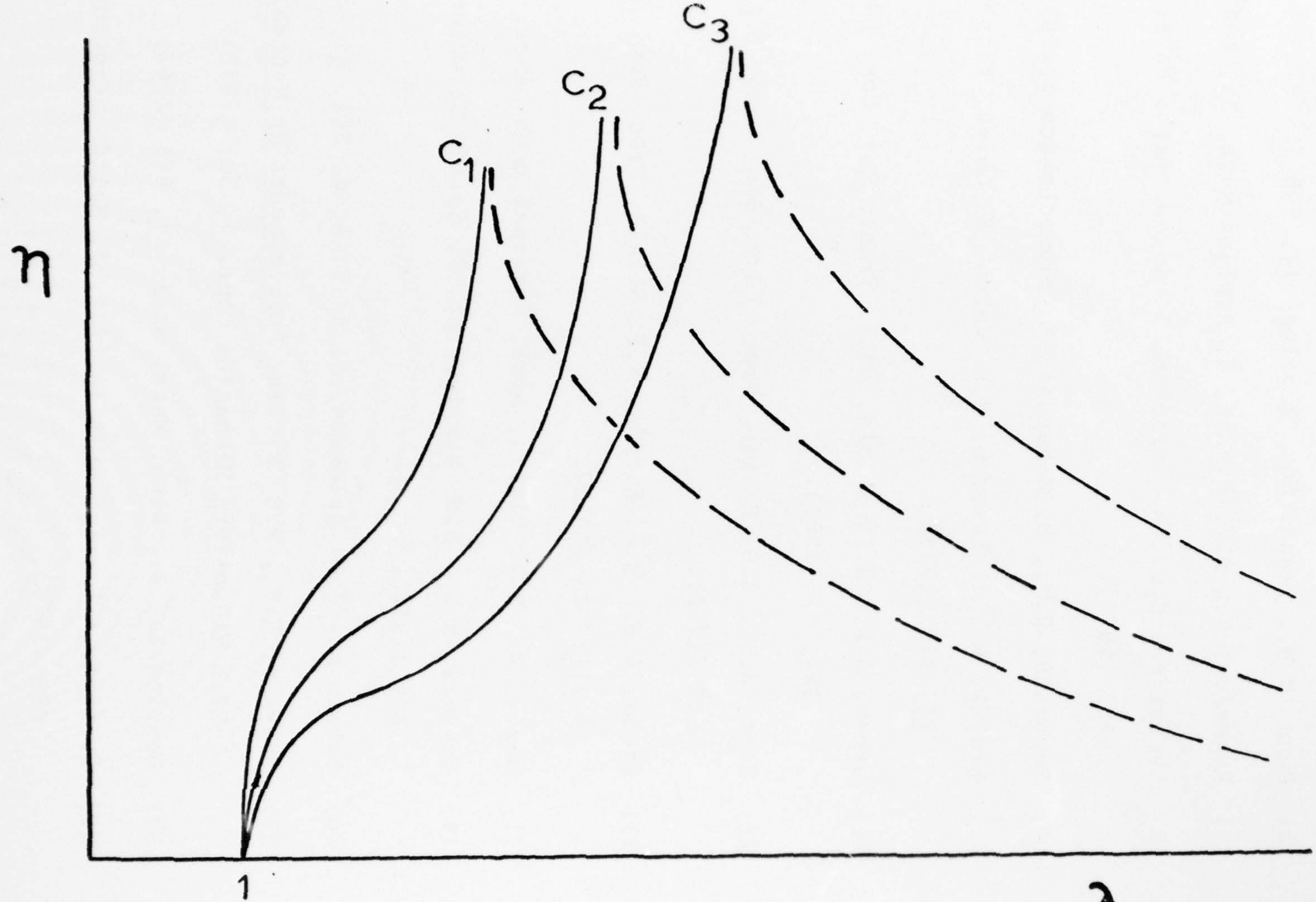

Fig.1. Schematic relations between λ and the critical value of λ for flexural bifurcation (full lines) and barreling bifurcation (broken lines) for three values of $A (C_3 > C_2 > C_1 > -1)$.

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